

# First Passage Under Restart

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First passage under restart has recently emerged as a conceptual framework suitable for the description of a wide range of phenomena, but the endless variety of ways in which restart mechanisms and first passage processes mix and match hindered the identification of unifying principles and general truths. Hope that these exist came from a recently discovered universality displayed by processes under optimal, constant rate, restart—but extensions and generalizations proved challenging as they marry arbitrarily complex processes and restart mechanisms. To address this challenge, we develop a generic approach to first passage under restart. Key features of diffusion under restart—the ultimate poster boy for this wide and diverse class of problems—are then shown to be completely universal.

A myriad of basic questions and a wide array of applications have turned first passage time (FPT) processes into a long standing focal point of scientific interest [1, 2]. These processes were studied extensively, e.g. in the context of nonequilibrium systems [3], but despite many years of study paramount discoveries are still being made and exciting applications continue to be found. Recently, several groups have observed that any FPT process imaginable can become subject to restart, i.e., can be stopped and started anew (Fig. 1). This observation has opened a rapidly moving theoretical research front [4–19] and applications to search problems [20–22], the optimization of randomized computer algorithms [23–28], and in the field of biophysics [30, 31], have further propelled its expansion. Universality has always been considered a holy grail of the physical sciences and novel revelations concerning universality in FPT processes have recently taken center stage and attracted considerable attention [32–34]. In contrast, not a lot is known in general about the problem of first passage under restart (FPUR).

Diffusion with resetting to the origin is a quintessential example of FPUR [5]. In this problem, a particle undergoes diffusion but from time to time is also taken and returned to the place from where it started its motion (reset or restart). In addition, at some distance away from the origin a target awaits and one is interested in the time it takes the particle to first get to the target, i.e., in its distribution and corresponding moments. This problem was first studied with restart rates that are constant in time and the surprise came from the fact that restart was able to expedite search and that a carefully chosen (optimal) restart rate could minimize the mean FPT to the target. Further down the road, other restart mechanisms were also studied [12, 13, 16, 19] and it was shown that these may under- or over-preform when compared to restart at a constant rate [12, 19].

Each variant above carried with it some unique and intriguing features, but exhausting the vast combinatorial space of process-restart pairs—one problem at a time—is

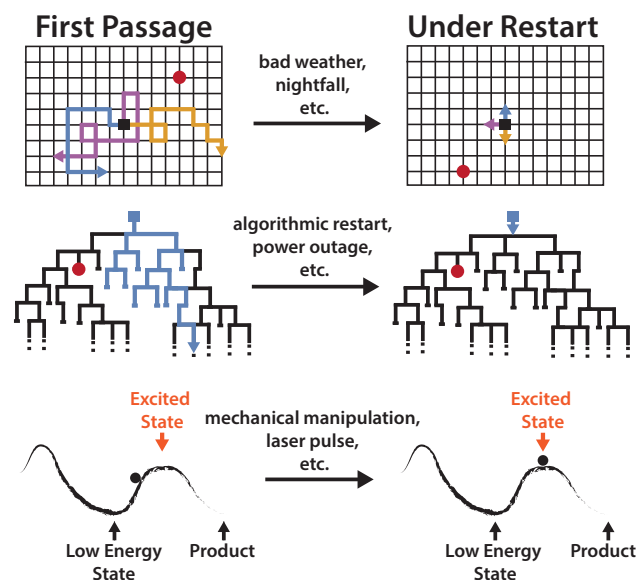


Figure 1: **Top.** Bad weather could force a team of searchers to temporarily cease their efforts and return to base. By the time search is renewed the target may have relocated and search must thus start from scratch. **Middle.** A computer algorithm operates as a black box which randomly scans a tree of possibilities in search of a solution. Chance may send the algorithm down the wrong path but programmed restart could help rescue the search. **Bottom.** A molecule that was previously prepared at an excited state decays to a low energy state. A pulse of laser could bring the molecule back to its excited state and restart a chemical or physical reaction. This time, the desired product may be formed.

virtually impossible. Indeed, restart processes may take different shapes and forms and the effect they have on FPT processes other than diffusion [35–42] is also of interest. Moreover, it is often the case—in real life scenarios—that the process under consideration, the restart mechanism that accompanies it, or both are poorly specified or even completely unknown. General approaches,

better suited to deal with partial and missing information and with the need to generalize from specific examples, could then become handy.

Recently, two attempts to unify treatment were made. In [18], an approach suitable to the description of a generic FPT process under constant rate restart was presented. The approach was utilized to show that when restart is optimal—the relative fluctuation in the FPT of the restarted process is always unity. This result holds true regardless of the underlying process, be it diffusion or other, but is no longer valid for time dependent restart rates as these were not covered by the approach to begin with. Restart rates with arbitrary time dependence were considered in [12], but analysis there was limited to diffusion and did not cover other FPT processes. Here, we will be interested in merging the two approaches in attempt to get the best of both worlds. To this end, we consider a generic FPT process that has further become subject to a generic restart mechanism. This setting is extremely general and captures, as special cases, the overwhelming majority of models that have already appeared in the literature. We analyze this scheme to attain, and concisely describe, several broad scope results which unravel universal features of this wide class of problems. In what follows, we use  $f_Z(t)$ ,  $\langle Z \rangle$ ,  $\sigma^2(Z)$  and  $\tilde{Z}(s) \equiv \langle e^{-sZ} \rangle$  to denote, respectively, the probability density function, expectation, variance, and Laplace transform of a real-valued random variable  $Z$ .

**Mean FPT under restart.** Consider a generic process that starts at time zero and, if allowed to take place without interruptions, ends after a random time  $T$ . The process is, however, restarted at some random time  $R$ . Thus, if the process is completed prior to restart the story there ends. Otherwise, the process will start from scratch and begin completely anew. This procedure repeats itself until the process reaches completion. Denoting the random completion time of the restarted process by  $T_R$  it can be seen that

$$T_R = \begin{cases} T & \text{if } T < R \\ R + T'_R & \text{if } R \leq T, \end{cases} \quad (1)$$

where  $T'_R$  is an independent and identically distributed copy of  $T_R$ .

A scheme similar to the one described in Eq. (1) was analyzed in [18]. There, no assumptions were made on the distribution of the time  $T$  which governs the completion of the underlying process, but the restart time  $R$  was assumed to be exponentially distributed with rate parameter  $r$ . This means that restart is conducted at a constant rate  $r$ , i.e., that for any given time point the probability that restart will occur at the next infinitesimal time interval  $dt$  is  $r dt$ . Here, we relax this assumption allowing for generally distributed restart times or, equivalently, for restart rates with arbitrary time dependence. Letting  $r(t)$  denote the restart rate at time  $t$ , we

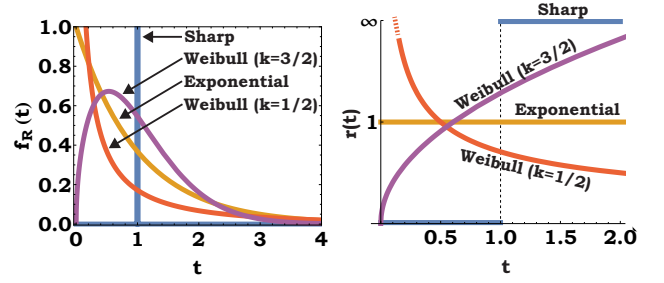


Figure 2: A few examples of restart time distributions (left) and the restart rates they induce (right). Below  $\delta(x)$  is the Dirac delta function,  $\Gamma(x)$  is the Gamma function, and  $\langle R \rangle = 1$  in all plots: (i) Deterministic (sharp) restart  $f_R(t) = \delta(t - \langle R \rangle)$ . Restart rate jumps abruptly from zero to infinity at  $t = \langle R \rangle$ ; (ii) Exponentially distributed restart  $f_R(t) = \langle R \rangle^{-1} e^{-t/\langle R \rangle}$ . Restart rate is constant:  $r(t) = 1/\langle R \rangle$ ; (iii & iv) Weibull distributed restart  $f_R(t) = \frac{k}{\lambda} \left(\frac{t}{\lambda}\right)^{k-1} e^{-(t/\lambda)^k}$ . Restart rate is given by  $r(t) = kt^{k-1}/\lambda^k$  and could monotonically decrease (e.g.  $k = 1/2$ ,  $\lambda = \langle R \rangle/2$ ) or increase (e.g.  $k = 3/2$ ,  $\lambda = \langle R \rangle/\Gamma(5/3)$ ) with time.

note that the two perspectives are related via (Fig. 2)

$$\Pr(R \leq t) = 1 - \exp\left(-\int_0^t r(x) dx\right), \quad (2)$$

where  $\Pr(R \leq t)$  is the probability that  $R \leq t$  [43].

Equation (1) could be used to provide a simple formula for the mean FPT of a stochastic process under restart. Indeed, noting that it can also be written as  $T_R = \min(T, R) + I\{R \leq T\}T'_R$ , where  $\min(T, R)$  is the minimum of  $T$  and  $R$  and  $I\{R \leq T\}$  is an indicator random variable which takes the value one when  $R \leq T$  and zero otherwise, we take expectations to find

$$\langle T_R \rangle = \frac{\langle \min(T, R) \rangle}{\Pr(T < R)}. \quad (3)$$

The right hand side of Eq. (3) can then be computed given the distributions of  $T$  and  $R$  if one also recalls that the cumulative distribution function of  $\min(T, R)$  is given by  $\Pr(\min(T, R) \leq t) = 1 - \Pr(T > t)\Pr(R > t)$ .

A hallmark of restart is its ability to minimize (optimize) mean FPTs. For example, when the restart rate  $r(t) = r$  is constant it is straight forward to show that Eq. (3) reduces to  $\langle T_R \rangle = (1 - \tilde{T}(r)) / (r\tilde{T}(r))$ , where  $\tilde{T}(r)$  is the Laplace transform of  $T$  evaluated at  $r$ . One could then seek an optimal rate  $r^*$  which brings  $\langle T_R \rangle$  to a minimum, derive general conditions for this rate to be strictly larger than zero, and further discuss universal properties of the optimal rate itself [15, 30]. Clearly, this line of inquiry is not limited to the case of exponentially distributed restart times and could also be extended to other parametric distributions. Various optimization questions could then be addressed directly, but we would now like to consider a broader optimization

question. Specifically, we ask if within the vast space of stochastic restart strategies, and irrespective of the underlying process being restarted, there is a single winning strategy that could not be beat?

**Sharp restart is a dominant strategy.** Consider a particle “searching” for a stationary target via one dimensional diffusion. The particle starts at the origin, the initial distance between the particle and the target is  $L$ , and the diffusion coefficient of the particle is  $D$ . Denoting the particle’s FPT to the target with  $T$ , the latter is known to come from the Lévy-Smirnov distribution  $f_T(t) = \sqrt{L^2/4D\pi t^3} e^{-L^2/4Dt}$  [1]. Considering the same problem under restart, we take  $D = 1/2$  and  $L = 1$ , and utilize Eq. (3) to plot  $\langle T_R \rangle$  as a function of  $\langle R \rangle$  for various restart time distributions (Fig. 3). As can be seen, a minimum of  $\langle T_R \rangle$  is always attained and while the values taken by the different minima and their positions clearly depend on the distribution of the restart time—it is sharp restart that attains the lowest of minima. A similar observation was made in the past and it was consequently conjectured that in the case of diffusion mediated search sharp restart is the optimal restart strategy [12, 19]. Strikingly, this is also true in general.

Consider, for the sake of simplicity, a random restart time  $R$  characterized by a proper density  $f_R(t)$  and note that  $\langle \min(T, R) \rangle = \int_0^\infty f_R(t) \langle \min(T, R) | R = t \rangle dt = \int_0^\infty f_R(t) \langle \min(T, t) \rangle dt$ , which then implies  $\langle T_R \rangle = \int_0^\infty \frac{f_R(t) Pr(T < t)}{Pr(T < R)} \langle \min(T, t) \rangle dt$ . However,  $\int_0^\infty \frac{f_R(t) Pr(T < t)}{Pr(T < R)} dt = 1$ , and  $\langle \min(T, t) \rangle / Pr(T < t)$  is simply the mean completion time of a process that is restarted sharply after  $t$  units of time. Thus, if there exists some  $t^*$  such that  $\frac{\langle \min(T, t^*) \rangle}{Pr(T < t^*)} \leq \frac{\langle \min(T, t) \rangle}{Pr(T < t)}$  for all  $t \geq 0$ —sharp restart at  $t^*$  will also beat any random restart time that is governed by a proper density. Moreover, the law of total expectation implies  $\langle \min(T, R) \rangle = \langle \langle \min(T, R) | R \rangle_T \rangle_R$  and steps similar to those taken above assert that (SI)

$$\frac{\langle \min(T, t^*) \rangle}{Pr(T < t^*)} \leq \frac{\langle \min(T, R) \rangle}{Pr(T < R)}, \quad (4)$$

for any random restart time  $R$  regardless of its distribution. Equation (4) thus asserts that sharp restart is optimal among all possible stochastic restart strategies in continuous time, and we refer the reader to Luby *et al.* for complementary, algorithm oriented, discussion on the discrete time case [23].

**Distribution of FPT under restart.** So far, we have only been concerned with the *mean* FPT of a restarted process but will now move on to discuss the full distribution of  $T_R$ . The scheme described in Eq. (1) suggests a direct approach for numerical simulation of FPUR (Fig. 4 left). In this approach, one draws two random times from the distributions of  $T$  and  $R$ , and only then—based on the outcome of that draw—decides which of the two, restart or completion, happened first.

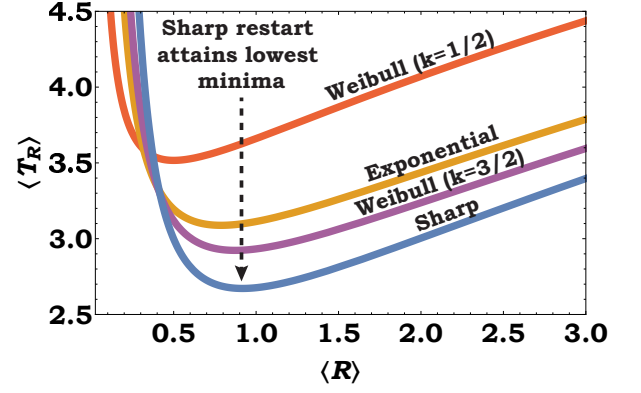


Figure 3: Mean FPT for diffusion mediated search with restart vs. the mean restart time, for various restart time distributions taken from Fig. 2.

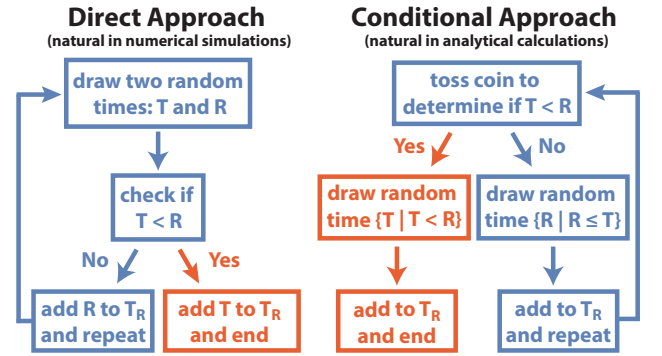


Figure 4: Two approaches to first passage under restart.

An equivalent approach would operate in reversed order. A coin with probability  $Pr(T < R)$  will first be tossed to determine if completion preceded restart (or vice versa) and only then, given that information, the appropriate—conditional—random time will be drawn (Fig. 4 right). This approach is somewhat awkward and indirect for the purpose of numerical simulations, but is actually quite natural when coming to compute expectations and Laplace transforms where one usually starts by conditioning on the occurrence of an event of interest. Indeed, analytical formulas could be simplified with the aid of two auxiliary random variables:  $R_{min} \equiv \{R | R = \min(R, T)\}$  and  $T_{min} \equiv \{T | T = \min(R, T)\}$ . In words,  $R_{min}$  is the random restart time given that restart occurred prior to completion, and  $T_{min}$  is defined in a similar manner. Conditioning on whether  $T < R$  and applying the law of total expectation to  $\tilde{T}_R(s) = \langle e^{-sT_R} \rangle$ , we obtain (SI)

$$\tilde{T}_R(s) = \frac{Pr(T < R) \tilde{T}_{min}(s)}{1 - Pr(R \leq T) \tilde{R}_{min}(s)}. \quad (5)$$

Equation (5) allows one to explicitly compute the distribution of  $T_R$  in Laplace space. For example, when  $T$  and  $R$  are correspondingly governed by probabil-

ity densities  $f_T(t)$  and  $f_R(t)$ , we have  $Pr(T < R) = \int_0^\infty f_T(t) (\int_t^\infty f_R(t') dt') dt$  and the probability densities governing  $T_{min}$  and  $R_{min}$  are similarly given by

$$\begin{aligned} f_{T_{min}}(t) &= f_T(t) \int_t^\infty f_R(t') dt' / Pr(T < R), \\ f_{R_{min}}(t) &= f_R(t) \int_0^t f_T(t') dt' / Pr(R \leq T). \end{aligned} \quad (6)$$

Plugging in concrete probability distributions explicit formulas can be obtained, e.g. for exponentially distributed restart  $f_R(t) = re^{-rt}$  and one could readily show that  $\tilde{T}_R(s) = \tilde{T}(s+r) / \left( \frac{s}{s+r} + \frac{r}{s+r} \tilde{T}(s+r) \right)$  (SI) as was previously obtained in [18] by other means.

**Fluctuations in FPT under optimal sharp restart obey a universal inequality.** Given Eq. (5), one could utilize the known relation between moments and the Laplace transform [41] to find (SI)

$$\langle T_R^2 \rangle = \frac{\langle \min(T, R)^2 \rangle}{Pr(T < R)} + \frac{2Pr(R \leq T) \langle R_{min} \rangle \langle \min(T, R) \rangle}{Pr(T < R)^2}. \quad (7)$$

A special case of this result was used to show that the relative fluctuation,  $\sigma(T_R) / \langle T_R \rangle$ , is always unity when a process is restarted at a constant rate  $r^* > 0$  which brings  $\langle T_R \rangle$  to a minimum. Optimal *sharp restart* could lower the mean FPT,  $\langle T_R \rangle$ , well below the value it attains for optimal *constant rate restart* but unless the resulting fold reduction is also matched or exceeded by a fold reduction in  $\sigma(T_R)$ —the relative fluctuation in the FPT would surely increase. It is thus possible that the ability of the sharp restart strategy to attain lower mean FPTs comes at the expense of higher relative fluctuations—and hence greater uncertainty—in the FPT itself. However, when Eq. (7) was utilized to examine diffusion and other case studies (Fig. 5), we consistently found

$$\sigma(T_{t^*}) / \langle T_{t^*} \rangle \leq 1, \quad (8)$$

for the relative fluctuation at the optimal restart time  $t^*$ .

Equation (8) is universal. To see this, we assume by contradiction that there exists a FPT process for which  $\sigma(T_{t^*}) / \langle T_{t^*} \rangle > 1$ ; and consider a restart strategy  $R_{mix}$  in which this process is restarted at a low constant rate  $r \ll 1$  in addition to being sharply restarted whenever a time  $t^*$  passes from the previous restart (or start) epoch. Applying this restart strategy is equivalent to augmenting the process under sharp restart with an additional restart mechanism that restarts it with rate  $r$ . However, if the relative fluctuation in the FPT of a process is larger than unity—restart at a low constant rate will surely lower its mean FPT (and vice versa). This is true regardless of the underlying process, and can be seen by examining  $\langle T_R \rangle$  for general  $T$ , and  $R$  which is exponentially distributed with rate  $r$  (see formula below Eq. (3)). Utilizing the moment representation of the Laplace transform, one can then show that  $[d\langle T_R \rangle / dr]_{r=0} < 0$  whenever  $\sigma(T) / \langle T \rangle > 1$  (SI). Denoting the mean FPT under

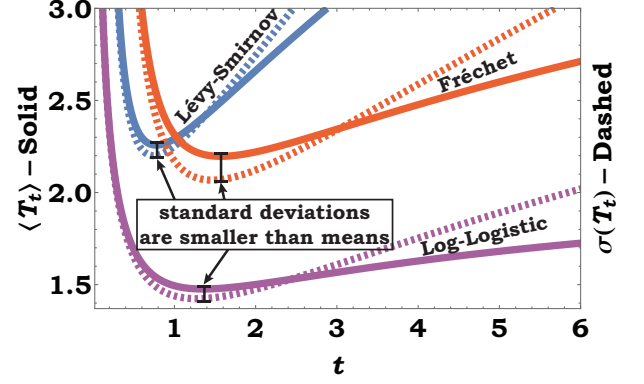


Figure 5: The mean (solid) and standard deviation (dashed) of the restarted FPT  $T_t$  vs. the sharp restart time  $t$ , for various distributions of the underlying FPT  $T$  (see SI for details).

$R_{mix}$  by  $\langle T_{R_{mix}} \rangle$ , and letting  $T_{t^*}$  take  $T$ 's place above, it follows that  $\langle T_{t^*} \rangle > \langle T_{R_{mix}} \rangle$ . We have thus found a non-sharp restart strategy which lowers the mean FPT beyond that attained for optimal sharp restart. However, this finding must be false as it stands in contradiction to the proven dominance of optimal sharp restart (discussion above), and Eq. (8) then follows immediately. More generally, an equation similar to Eq. (8) must hold for every restart strategy  $R$  which attains a FPT that cannot be lowered further by introducing an additional restart rate  $r \ll 1$ , and Eqs. (3) and (7) could then be utilized to comprehensively characterize this set of strategies (SI)

$$\sigma(T_R) / \langle T_R \rangle \leq 1 \iff \langle T_{min} \rangle \geq \frac{1}{2} \frac{\langle \min(T, R)^2 \rangle}{\langle \min(T, R) \rangle}. \quad (9)$$

A probabilistic interpretation of this result and discussion with examples are given in the SI.

**Conclusions and outlook.** In this letter we developed a theoretical framework for first passage under restart. With its aid, we showed how simple observations made for diffusion under restart can be elevated to the level of generic statements which capture fundamental aspects of the phenomena. The universal dominance of sharp restart over other restart strategies is noteworthy. However, while this strategy can be readily applied in some settings, its realization in others may require going to extremes. Particularly, in biophysical settings the generation of tight time distributions relies on the concatenation of irreversible molecular transitions. Restart plays a role in such systems [30, 31], but the energetic cost associated with creating an (almost) irreversible transition, and the infinitely many required for *mathematically sharp* restart, would surely give rise to interesting trade-offs. The incorporation of such thermodynamic considerations into the framework presented herein à la [44, 45], and the identification of those nearly optimal strategies (non-sharp but punctual) [46] which perform best under energy consumption constraints, is yet a future challenge.



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- [1] Redner, S., 2001. A guide to first-passage processes. Cambridge University Press.
  - [2] Metzler, R., Redner, S. and Oshanin, G., 2014. First-Passage Phenomena and Their Applications (Vol. 35). Singapore: World Scientific.
  - [3] Bray, A.J., Majumdar, S.N. and Schehr, G., 2013. Persistence and first-passage properties in nonequilibrium systems. *Advances in Physics*, 62(3), pp.225-361.
  - [4] Manrubia, S.C. and Zanette, D.H., 1999. Stochastic multiplicative processes with reset events. *Physical Review E*, 59(5), p.4945.
  - [5] Evans, M.R. and Majumdar, S.N., 2011. Diffusion with stochastic resetting. *Physical review letters*, 106(16), p.160601.
  - [6] Evans, M.R. and Majumdar, S.N., 2011. Diffusion with optimal resetting. *Journal of Physics A: Mathematical and Theoretical*, 44(43), p.435001.
  - [7] Montero, M. and Villarroel, J., 2013. Monotonic continuous-time random walks with drift and stochastic reset events. *Physical Review E*, 87(1), p.012116.
  - [8] Durang, X., Henkel, M. and Park, H., 2014. The statistical mechanics of the coagulation–diffusion process with a stochastic reset. *Journal of Physics A: Mathematical and Theoretical*, 47(4), p.045002.
  - [9] Gupta, S., Majumdar, S.N. and Schehr, G., 2014. Fluctuating interfaces subject to stochastic resetting. *Physical review letters*, 112(22), p.220601.
  - [10] Pal, A., 2015. Diffusion in a potential landscape with stochastic resetting. *Physical Review E*, 91(1), p.012113.
  - [11] Majumdar, S.N., Sabhapandit, S. and Schehr, G., 2015. Random walk with random resetting to the maximum position. *Physical Review E*, 92(5), p.052126.
  - [12] Pal, A., Kundu, A. and Evans, M.R., 2016. Diffusion under time-dependent resetting. *Journal of Physics A: Mathematical and Theoretical*, 49(22), p.225001.
  - [13] Nagar, A. and Gupta, S., 2016. Diffusion with stochastic resetting at power-law times. *Physical Review E*, 93(6), p.060102.
  - [14] Meylahn, J.M., Sabhapandit, S. and Touchette, H., 2015. Large deviations for Markov processes with resetting. *Physical Review E*, 92(6), p.062148.
  - [15] Rotbart, T., Reuveni, S. and Urbakh, M., 2015. Michaelis-Menten reaction scheme as a unified approach towards the optimal restart problem. *Physical Review E*, 92(6), p.060101.
  - [16] Eule, S. and Metzger, J.J., 2016. Non-equilibrium steady states of stochastic processes with intermittent resetting. *New Journal of Physics*, 18(3), p.033006.
  - [17] Fuchs, J., Goldt, S. and Seifert, U., 2016. Stochastic thermodynamics of resetting. *EPL (Europhysics Letters)*, 113(6), p.60009.
  - [18] Reuveni, S., 2016. Optimal stochastic restart renders fluctuations in first passage times universal. *Physical review letters*, 116(17), p.170601.
  - [19] Bhat, U., De Bacco, C. and Redner, S., 2016. Stochastic Search with Poisson and Deterministic Resetting. *arXiv preprint arXiv:1605.08812*.
  - [20] Eliazar, I., Koren, T. and Klafter, J., 2007. Searching circular DNA strands. *Journal of Physics: Condensed Matter*, 19(6), p.065140.
  - [21] Kusmierz, L., Majumdar, S.N., Sabhapandit, S. and Schehr, G., 2014. First order transition for the optimal search time of Lévy flights with resetting. *Physical review letters*, 113(22), p.220602.
  - [22] Kuśmierz, Ł. and Gudowska-Nowak, E., 2015. Optimal first-arrival times in Lévy flights with resetting. *Physical Review E*, 92(5), p.052127.
  - [23] Luby, M., Sinclair, A. and Zuckerman, D., 1993, June. Optimal speedup of Las Vegas algorithms. In *Theory and Computing Systems*, 1993., Proceedings of the 2nd Israel Symposium on the (pp. 128-133). IEEE.
  - [24] Gomes, C.P., Selman, B. and Kautz, H., 1998. Boosting combinatorial search through randomization. *AAAI/IAAI*, 98, pp.431-437.
  - [25] Walsh, T., 1999, July. Search in a small world. In *IJCAI (Vol. 99, pp. 1172-1177)*.
  - [26] Moskewicz, M.W., Madigan, C.F., Zhao, Y., Zhang, L. and Malik, S., 2001, June. Chaff: Engineering an efficient SAT solver. In *Proceedings of the 38th annual Design Automation Conference (pp. 530-535)*. ACM.
  - [27] Montanari, A. and Zecchina, R., 2002. Optimizing searches via rare events. *Physical review letters*, 88(17), p.178701.
  - [28] Huang, J., 2007, January. The Effect of Restarts on the Efficiency of Clause Learning. In *IJCAI (Vol. 7, pp. 2318-2323)*.
  - [29] Steiger, D.S., Rønnow, T.F. and Troyer, M., 2015. Heavy Tails in the Distribution of Time to Solution for Classical and Quantum Annealing. *Physical review letters*, 115(23), p.230501.
  - [30] Reuveni, S., Urbakh, M. and Klafter, J., 2014. Role of substrate unbinding in Michaelis–Menten enzymatic reactions. *Proceedings of the National Academy of Sciences*, 111(12), pp.4391-4396.
  - [31] Roldán, É., Lisica, A., Sánchez-Taltavull, D. and Grill, S.W., 2016. Stochastic resetting in backtrack recovery by RNA polymerases. *Physical Review E*, 93, 062411 (2016).
  - [32] Condamin, S., Bénichou, O., Tejedor, V., Voituriez, R. and Klafter, J., 2007. First-passage times in complex scale-invariant media. *Nature*, 450(7166), pp.77-80.
  - [33] Bénichou, O., Chevalier, C., Klafter, J., Meyer, B. and Voituriez, R., 2010. Geometry-controlled kinetics. *Nature chemistry*, 2(6), pp.472-477.
  - [34] Chupeau, M., Bénichou, O. and Voituriez, R., 2015. Cover times of random searches. *Nature Physics*.
  - [35] Lomholt, M.A., Tal, K., Metzler, R. and Joseph, K., 2008. Lévy strategies in intermittent search processes are advantageous. *Proceedings of the National Academy of Sciences*, 105(32), pp.11055-11059.
  - [36] Bénichou, O., Loverdo, C., Moreau, M. and Voituriez, R., 2011. Intermittent search strategies. *Reviews of Modern Physics*, 83(1), p.81.
  - [37] Palyulin, V.V., Chechkin, A.V. and Metzler, R., 2014. Lévy flights do not always optimize random blind search for sparse targets. *Proceedings of the National Academy*

- of Sciences, 111(8), pp.2931-2936.
- [38] Metzler, R. and Klafter, J., 2000. The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Physics reports*, 339(1), pp.1-77.
  - [39] Sokolov, I.M., Klafter, J. and Blumen, A., 2002. Fractional kinetics. *Physics Today*, 55(11), pp.48-54.
  - [40] Reuveni, S., Granek, R. and Klafter, J., 2010. Vibrational shortcut to the mean-first-passage-time problem. *Physical Review E*, 81(4), p.040103.
  - [41] Klafter, J. and Sokolov, I.M., 2011. *First steps in random walks: from tools to applications*. Oxford University Press.
  - [42] Iyer-Biswas, S. and Zilman, A., 2016. First-Passage Processes in Cellular Biology. *Advances in Chemical Physics*, Volume 160, pp.261-306.
  - [43] Barlow, Richard E., and Frank Proschan. *Mathematical Theory of Reliability*. Vol. 17. SIAM, 1996.
  - [44] Barato, A.C. and Seifert, U., 2015. Thermodynamic uncertainty relation for biomolecular processes. *Physical review letters*, 114(15), p.158101.
  - [45] Barato, A.C. and Seifert, U., 2015. Universal bound on the Fano factor in enzyme kinetics. *The Journal of Physical Chemistry B*, 119(22), pp.6555-6561.
  - [46] Husain, K. and Krishna, S., 2016. Efficiency of a Stochastic Search with Punctual and Costly Restarts. *arXiv preprint arXiv:1609.03754*.

# Supplementary Information

## First Passage Under Restart

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## Contents

<b>1</b>	<b>Derivation of Eq. (4) in main text</b>	<b>3</b>
<b>2</b>	<b>Derivation of Eq. (5) in main text</b>	<b>3</b>
2.1	The case of exponential restart times . . . . .	4
<b>3</b>	<b>Derivation of Eq. (7) in main text</b>	<b>5</b>
<b>4</b>	<b>Derivation of Eq. (8) in main text</b>	<b>7</b>
<b>5</b>	<b>Details of distributions in Fig. 5</b>	<b>7</b>
<b>6</b>	<b>Derivation and probabilistic interpretation of Eq. (9) in main text</b>	<b>8</b>
<b>7</b>	<b>Numerical exploration of Eq. (9) in the main text</b>	<b>9</b>



## 1 Derivation of Eq. (4) in main text

The same line of argumentation brought in the text could also be used to show that the sharp restart strategy,  $Pr(R = t^*) = 1$ , is optimal among all possible stochastic restart strategies. Indeed, the law of total expectation asserts that

$$\langle \min(T, R) \rangle = \langle \langle \min(T, R) | R \rangle_T \rangle_R, \quad (1.1)$$

where  $\langle \cdot \rangle_R$  and  $\langle \cdot \rangle_T$  explicitly mark the expectation with respect to  $R$  and  $T$ . Utilizing Eq. (3) in the text, and letting  $I\{T < R\} = 1 - I\{T \geq R\}$  denote an indicator random variable which takes the value one when  $T < R$  and zero otherwise, it follows that

$$\begin{aligned} \langle T_R \rangle &= \frac{\langle \min(T, R) | R \rangle_T \rangle_R}{Pr(T < R)} = \frac{\langle \frac{Pr(T < R | R)}{Pr(T < R | R)} \langle \min(T, R) | R \rangle_T \rangle_R}{\langle I\{T < R\} \rangle_{R, T}} \\ &= \left\langle \frac{Pr(T < R | R)}{\langle I\{T < R\} \rangle_{R, T}} \frac{\langle \min(T, R) | R \rangle_T}{Pr(T < R | R)} \right\rangle_R. \end{aligned} \quad (1.2)$$

However, note that since

$$\left\langle \frac{Pr(T < R | R)}{\langle I\{T < R\} \rangle_{R, T}} \right\rangle_R = \frac{\langle I\{T < R\} | R \rangle_T \rangle_R}{\langle I\{T < R\} \rangle_{R, T}} = 1, \quad (1.3)$$

if an optimal sharp restart time  $R = t^*$  exists, i.e., one which satisfies

$$\begin{aligned} \frac{\langle \min(T, t^*) \rangle}{Pr(T < t^*)} &= \frac{\langle \min(T, R) | R=t^* \rangle_T}{Pr(T < R | R=t^*)} \\ &\leq \frac{\langle \min(T, R) | R=t \rangle_T}{Pr(T < R | R=t)} = \frac{\langle \min(T, t) \rangle}{Pr(T < t)}, \end{aligned} \quad (1.4)$$

for all  $t > 0$ , then using Eqs. (1.2) and (1.4) we have

$$\frac{\langle \min(T, t^*) \rangle}{Pr(T < t^*)} \leq \frac{\langle \min(T, R) \rangle}{Pr(T < R)} = \langle T_R \rangle, \quad (1.5)$$

for every stochastic restart time  $R$  and regardless of its distribution.

## 2 Derivation of Eq. (5) in main text

To derive Eq. (5) in the main text we note that

$$\begin{aligned} \tilde{T}_R(s) &= \langle e^{-sT_R} \rangle = Pr(T < R) \langle e^{-sT_R} | T < R \rangle \\ &\quad + Pr(R \leq T) \langle e^{-sT_R} | R \leq T \rangle, \end{aligned} \quad (2.1)$$

which gives

$$\begin{aligned} \tilde{T}_R(s) &= Pr(T < R) \langle e^{-s\{T_R | T < R\}} \rangle \\ &\quad + Pr(R \leq T) \langle e^{-s\{T_R | R \leq T\}} \rangle. \end{aligned} \quad (2.2)$$

However, utilizing Eq. (1) in the main text and recalling the way  $R_{min}$  and  $T_{min}$  were defined, we see that

$$\begin{aligned}\{T_R | T < R\} &= \{T | T < R\} \\ &= \{T | T = \min(R, T)\} = T_{min},\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}\{T_R | R \leq T\} &= \{R + T'_R | R \leq T\} \\ &= \{R | R = \min(R, T)\} + T'_R = R_{min} + T'_R,\end{aligned}\tag{2.4}$$

where in the second transition in Eq. (2.4) we have further used the fact that  $T'_R$  is an independent and identically distributed copy of  $T_R$  and hence independent of both  $R$  and  $T$ . We thus have

$$\begin{aligned}\tilde{T}_R(s) &= Pr(T < R) \langle e^{-sT_{min}} \rangle \\ &\quad + Pr(R \leq T) \langle e^{-s(R_{min} + T'_R)} \rangle \\ &= Pr(T < R) \tilde{T}_{min}(s) \\ &\quad + Pr(R \leq T) \tilde{R}_{min}(s) \tilde{T}_R(s),\end{aligned}\tag{2.5}$$

where in the last step we have again used the fact that  $T'_R$  is an independent and identically distributed copy of  $T_R$  and hence  $\langle e^{-s(R_{min} + T'_R)} \rangle = \langle e^{-sR_{min}} \rangle \langle e^{-sT'_R} \rangle = \langle e^{-sR_{min}} \rangle \langle e^{-sT_R} \rangle$ . Rearranging Eq. (2.5) we have

$$\tilde{T}_R(s) = \frac{Pr(T < R) \tilde{T}_{min}(s)}{1 - Pr(R \leq T) \tilde{R}_{min}(s)},\tag{2.6}$$

which identifies with Eq. (5) in the main text.

## 2.1 The case of exponential restart times

When the restart time  $R$  is exponentially distributed with rate  $r$  its probability density function is given by

$$f_R(t) = re^{-rt}.\tag{2.7}$$

The terms in Eq. (2.6) (Eq. (5) in the main text) can then be worked out to give

$$\begin{aligned}Pr(T < R) \tilde{T}_{min}(s) &= Pr(T < R) \langle e^{-s\{T|T < R\}} \rangle \\ &= Pr(T < R) \frac{\langle \int_0^\infty f_R(t) e^{-sT} dt \rangle_T}{Pr(T < R)} = \langle e^{-sT} \int_0^\infty re^{-rt} dt \rangle_T \\ &= \langle e^{-sT} e^{-rT} \rangle_T = \langle e^{-(s+r)T} \rangle_T = \tilde{T}(r + s),\end{aligned}\tag{2.8}$$

and

$$\begin{aligned}
Pr(R \leq T) \tilde{R}_{min}(s) &= Pr(R \leq T) \langle e^{-s\{R|R \leq T\}} \rangle \\
&= Pr(R \leq T) \frac{\langle \int_0^T f_R(t) e^{-st} dt \rangle_T}{Pr(R \leq T)} = \left\langle \int_0^T r e^{-rt} e^{-st} dt \right\rangle_T \\
&= \frac{r}{r+s} \langle 1 - e^{-(s+r)T} \rangle_T = \frac{r}{r+s} \left( 1 - \tilde{T}(r+s) \right).
\end{aligned} \tag{2.9}$$

Substituting back we find

$$\tilde{T}_R(s) = \frac{\tilde{T}(r+s)}{1 - \frac{r}{r+s} \left( 1 - \tilde{T}(r+s) \right)} = \frac{\tilde{T}(r+s)}{\frac{s}{r+s} + \frac{r}{r+s} \tilde{T}(r+s)}, \tag{2.10}$$

which identifies with the result in the main text.

### 3 Derivation of Eq. (7) in main text

To derive Eq. (7) in the main text, we first recall the relation between a Laplace transform of a random variable and its  $n$ -th moment

$$\langle Z^n \rangle = (-1)^n \frac{d^n \tilde{Z}(s)}{ds^n} \Big|_{s=0}. \tag{3.1}$$

Recalling Eq. (2.5) above

$$\tilde{T}_R(s) = Pr(T < R) \langle e^{-sT_{min}} \rangle + Pr(R \leq T) \langle e^{-s(R_{min} + T'_R)} \rangle, \tag{3.2}$$

we multiply both sides by -1, take a single derivative of with respect to  $s$ , and the limit of  $s \rightarrow 0$ , to obtain

$$\begin{aligned}
\langle T_R \rangle &= Pr(T < R) \langle T_{min} \rangle + Pr(R \leq T) \langle R_{min} + T'_R \rangle \\
&= Pr(T < R) \langle T_{min} \rangle + Pr(R \leq T) (\langle R_{min} \rangle + \langle T'_R \rangle).
\end{aligned} \tag{3.3}$$

This result is equivalent to Eq. (3) in the main text. Taking two derivatives of Eq. (3.2) with respect to  $s$  and the limit  $s \rightarrow 0$ , we find

$$\begin{aligned}
\langle T_R^2 \rangle &= Pr(T < R) \langle T_{min}^2 \rangle + Pr(R \leq T) \langle (R_{min} + T'_R)^2 \rangle \\
&= Pr(T < R) \langle T_{min}^2 \rangle + Pr(R \leq T) [\langle R_{min}^2 \rangle + \langle (T'_R)^2 \rangle + 2\langle R_{min} T'_R \rangle] \\
&= Pr(T < R) \langle T_{min}^2 \rangle + Pr(R \leq T) [\langle R_{min}^2 \rangle + \langle T_R^2 \rangle + 2\langle R_{min} \rangle \langle T_R \rangle].
\end{aligned} \tag{3.4}$$

Noting that

$$\begin{aligned}
\langle \min(T, R)^2 \rangle &= Pr(T < R) \langle T^2 | T < R \rangle + Pr(R \leq T) \langle R^2 | R \leq T \rangle \\
&= Pr(T < R) \langle T_{min}^2 \rangle + Pr(R \leq T) \langle R_{min}^2 \rangle,
\end{aligned} \tag{3.5}$$

using Eq. (3) in the main text, and rearranging terms in Eq. (3.4), we recover Eq. (7) in the main text

$$\langle T_R^2 \rangle = \frac{\langle \min(T, R)^2 \rangle}{Pr(T < R)} + \frac{2Pr(R \leq T) \langle R_{min} \rangle \langle \min(T, R) \rangle}{Pr(T < R)^2}. \quad (3.6)$$

An equivalent way in which Eq. (7) can be derived is by recalling that the Laplace transform of a random variable has the following moment expansion

$$\tilde{Z}(s) = \langle e^{-sZ} \rangle = 1 - s \langle Z \rangle + \frac{1}{2} s^2 \langle Z^2 \rangle + o(s^2). \quad (3.7)$$

On the one hand we thus have

$$\tilde{T}_R(s) = 1 - \langle T_R \rangle s + \frac{1}{2} \langle T_R^2 \rangle s^2 + o(s^2), \quad (3.8)$$

and from the other by use of Eq. (5) in the main text

$$\tilde{T}_R(s) = \frac{Pr(T < R) [1 - s \langle T_{min} \rangle + \frac{1}{2} s^2 \langle T_{min}^2 \rangle + o(s^2)]}{1 - Pr(R \leq T) [1 - s \langle R_{min} \rangle + \frac{1}{2} s^2 \langle R_{min}^2 \rangle + o(s^2)]}. \quad (3.9)$$

Expanding the right hand side of Eq. (3.9) to second order in the Laplace variable “ $s$ ” and equating coefficients of equal powers we find

$$\langle T_R \rangle = \frac{Pr(R \leq T) \langle R_{min} \rangle + Pr(T < R) \langle T_{min} \rangle}{Pr(T < R)} = \frac{\langle \min(T, R) \rangle}{Pr(T < R)}, \quad (3.10)$$

and

$$\begin{aligned} \langle T_R^2 \rangle &= 2 \left[ \langle R_{min} \rangle^2 + \frac{\langle R_{min} \rangle^2}{Pr(T < R)^2} - \frac{2 \langle R_{min} \rangle^2}{Pr(T < R)} - \langle R_{min} \rangle \langle T_{min} \rangle \right. \\ &\quad \left. + \frac{\langle R_{min} \rangle \langle T_{min} \rangle}{Pr(T < R)} - \frac{\langle R_{min}^2 \rangle}{2} + \frac{\langle R_{min}^2 \rangle}{2 Pr(T < R)} + \frac{\langle T_{min}^2 \rangle}{2} \right] \\ &= 2 \left[ \langle R_{min} \rangle^2 \frac{Pr(T \geq R)^2}{Pr(T < R)^2} + \langle R_{min} \rangle \langle T_{min} \rangle \frac{Pr(T \geq R)}{Pr(T < R)} \right. \\ &\quad \left. + \langle R_{min}^2 \rangle \frac{Pr(T \geq R)}{2 Pr(T < R)} + \frac{\langle T_{min}^2 \rangle}{2} \right] \end{aligned} \quad (3.11)$$

Equation (3.10) reaffirms Eq. (3) in the main text, rearranging Eq. (3.11) we find

$$\begin{aligned} \langle T_R^2 \rangle &= 2 \left[ \frac{\langle R_{min}^2 \rangle Pr(T \geq R) + \langle T_{min}^2 \rangle Pr(T < R)}{2 Pr(T < R)} \right. \\ &\quad \left. + \frac{\langle R_{min} \rangle Pr(T \geq R) (\langle R_{min} \rangle Pr(T \geq R) + \langle T_{min} \rangle Pr(T < R))}{Pr(T < R)^2} \right] \\ &= \frac{\langle \min(T, R)^2 \rangle}{Pr(T < R)} + \frac{2 Pr(T \geq R) \langle R_{min} \rangle \langle \min(T, R) \rangle}{Pr(T < R)^2}, \end{aligned} \quad (3.12)$$

which coincides with Eq. (7) in the main text.

## 4 Derivation of Eq. (8) in main text

To derive Eq. (8) in the main text we first show that when a process with first passage time  $T$ , such that  $\sigma(T)/\langle T \rangle > 1$ , is restarted at some low constant rate  $r$ , the mean first passage time of the restarted process will surely be smaller than  $\langle T \rangle$ . We start by multiplying both sides of Eq. (2.10) by  $-1$ , taking a single derivative with respect to  $s$ , and the limit of  $s \rightarrow 0$  to obtain<sup>1</sup>

$$\langle T_R \rangle = \left(1 - \tilde{T}(r)\right) / \left(r\tilde{T}(r)\right). \quad (4.1)$$

To probe the behavior of  $\langle T_R \rangle$  in the limit  $r \rightarrow 0$ , we expand  $\tilde{T}(r)$  to second order

$$\tilde{T}(r) = 1 - \langle T \rangle r + \frac{1}{2} \langle T^2 \rangle r^2 + o(r^2), \quad (4.2)$$

and substitute the result back into Eq. (4.1) to give

$$\langle T_R \rangle = \frac{\langle T \rangle r - \frac{1}{2} \langle T^2 \rangle r^2 - o(r^2)}{r - \langle T \rangle r^2 + \frac{1}{2} \langle T^2 \rangle r^3 + o(r^3)}. \quad (4.3)$$

It is now easy to see that

$$\begin{aligned} \langle T_R \rangle &= \langle T \rangle + \left[ \langle T \rangle^2 - \frac{1}{2} \langle T^2 \rangle \right] r + o(r^2) \\ &= \langle T \rangle + \frac{1}{2} \left[ \langle T \rangle^2 - \sigma^2(T) \right] r + o(r^2), \end{aligned} \quad (4.4)$$

which in turn means that when  $\sigma(T)/\langle T \rangle > 1$  and when  $r$  is sufficiently small

$$\langle T_R \rangle < \langle T \rangle. \quad (4.5)$$

In deriving Eq. (4.5) above we did not make any assumptions with regard to the origin or distribution of the first passage time  $T$  albeit the assumption that  $\sigma(T)/\langle T \rangle > 1$ . In principle,  $T$  could also be the first passage time of a process which is already subject to restart and even that of a process which is subject to *optimal sharp restart*. However, and as explained in the main text, the latter case is impossible as it would in turn mean that a non-sharp, or mixed, restart strategy can attain lower mean first passage times than those attained for optimal sharp restart.

## 5 Details of distributions in Fig. 5

Plot were made for the following distributions:

1. Lévy-Smirnov (FPT of diffusion mediated search)

$$f_T(t) = \sqrt{L^2/4D\pi t^3} e^{-L^2/4Dt}, \quad (5.1)$$

( $t \geq 0$ ), with  $\sqrt{L^2/4D} = 0.65$ .

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<sup>1</sup>Alternatively, Eq. (4.1) could also be obtained directly from from Eq. (3) in the main text by exploiting the fact that when the restart time  $R$  is exponential with rate  $r$  its probability density function is given by  $f_R(t) = re^{-rt}$ .

2. Fréchet

$$Pr(T \leq t) = e^{-t^{-\alpha}}, \quad (5.2)$$

( $t \geq 0$ ), with  $\alpha = 1$ .

3. Log-Logistic

$$Pr(T \leq t) = \left[1 + (t/\alpha)^{-\beta}\right]^{-1}, \quad (5.3)$$

( $t \geq 0$ ), with  $\alpha = 1$  and  $\beta = 3/2$ .

## 6 Derivation and probabilistic interpretation of Eq. (9) in main text

To derive Eq. (9) in the main text we first note that

$$\sigma(T_R) / \langle T_R \rangle \leq 1 \iff \langle T_R^2 \rangle \leq 2 \langle T_R \rangle^2. \quad (6.1)$$

Plugging in Eqs. (3) and (7) in the main text we write the right inequality in (6.1) explicitly

$$\frac{\langle \min(T, R)^2 \rangle}{Pr(T < R)} + \frac{2Pr(R \leq T) \langle R_{min} \rangle \langle \min(T, R) \rangle}{Pr(T < R)^2} \leq 2 \frac{\langle \min(T, R) \rangle^2}{Pr(T < R)^2}, \quad (6.2)$$

and rearrange to obtain

$$\frac{\langle \min(T, R)^2 \rangle}{2} \leq \frac{\langle \min(T, R) \rangle}{Pr(T < R)} [\langle \min(T, R) \rangle - Pr(R \leq T) \langle R_{min} \rangle]. \quad (6.3)$$

Recalling that

$$\langle \min(T, R) \rangle = Pr(T < R) \langle T_{min} \rangle + Pr(R \leq T) \langle R_{min} \rangle, \quad (6.4)$$

we see that Eq. (6.3) can be written as

$$\langle T_{min} \rangle \geq \frac{1}{2} \frac{\langle \min(T, R)^2 \rangle}{\langle \min(T, R) \rangle}, \quad (6.5)$$

which proves Eq. (9) in the main text.

The result in Eq. (9) could be understood probabilistically by considering the effect the introduction of a low restart rate,  $r \ll 1$ , has on the mean FPT,  $\langle T_R \rangle$ , of a process that is already subject to restart. In section (4) above we have shown that this would lower  $\langle T_R \rangle$  if  $\sigma(T_R) / \langle T_R \rangle > 1$ , but would increase  $\langle T_R \rangle$  (or leave it unchanged) otherwise. Equation (9) in the main text asserts that we can replace  $\sigma(T_R) / \langle T_R \rangle > 1$  in the previous sentence with  $\frac{1}{2} \frac{\langle \min(T, R)^2 \rangle}{\langle \min(T, R) \rangle} > \langle T_{min} \rangle$ . To better understand why, imagine that a low restart rate  $r \ll 1$  is introduced to the system from the outside and consider the effect this will have on the expected completion time of the process. The external restart rate added is infinitesimally small, but it will eventually find the process at some random point in time and restart it. The expected completion time from that point onward is  $\langle T_R \rangle$  (to an excellent approximation) as an additional, exogenous, restart event within this time frame is extremely unlikely. This expected time



to completion will now be compared to that which would have been attained in the absence of exogenous restart.

Immediately after our process has started the mean time taken for it to either complete or restart is given by  $\langle \min(T, R) \rangle$ . However, when the process is visited at some random point in time (after it has already started rolling) this is no longer the case. Indeed, one is then interested in the mean *residual time* (averaged over random points in time) that is left until either restart or completion happen, and for renewal processes this time is generally known to be given by [1]

$$\langle T_{res} \rangle = \frac{1}{2} \frac{\langle \min(T, R)^2 \rangle}{\langle \min(T, R) \rangle}. \quad (6.6)$$

And so, in the absence of exogenous restart, one would need to wait  $\langle T_{res} \rangle$  units of time (on average) for the process to either complete or restart (counting from that random point in time at which the process was visited). To that, one needs not add a thing if the process completes, or is required to add  $\langle T_R \rangle$  units of time (on average) if the processes restarts. What is, however, the probability that the latter happens?

To answer this, we once again observe that if we were to examine our process immediately after it has started, the answer would have been  $Pr(R \leq T)$ . However, we are now interested in the probability that  $R \leq T$  given that we observe the process at a random point in time and after it has already been given the opportunity to “age”. This probability is in turn given by  $\frac{Pr(R \leq T) \langle R_{min} \rangle}{\langle \min(T, R) \rangle} = \frac{Pr(R \leq T) \langle R_{min} \rangle}{Pr(T < R) \langle T_{min} \rangle + Pr(R \leq T) \langle R_{min} \rangle}$  which is the exact relative fraction, on the time axis, captured by those time spans which end with (endogenous) restart rather than completion. We thus find that the introduction of a low restart rate will increase the mean FPT (or leave it unchanged) whenever

$$\langle T_R \rangle \geq \langle T_{res} \rangle + \frac{Pr(R \leq T) \langle R_{min} \rangle}{\langle \min(T, R) \rangle} \langle T_R \rangle, \quad (6.7)$$

and would otherwise lower it. Plugging in the expressions for  $\langle T_{res} \rangle$  and  $\langle T_R \rangle$  into Eq. (6.7) we have

$$\frac{\langle \min(T, R) \rangle}{Pr(T < R)} \geq \frac{1}{2} \frac{\langle \min(T, R)^2 \rangle}{\langle \min(T, R) \rangle} + \frac{Pr(R \leq T) \langle R_{min} \rangle}{Pr(T < R)}. \quad (6.8)$$

and rearrangement then gives back Eq. (6.5).

## 7 Numerical exploration of Eq. (9) in the main text

To demonstrate the validity and generality of Eq. (9) in the main text we explore several numerical examples. In Figure (S1), we revisit the examples given in Fig. 5 (main text). There, the restart time distribution was sharp ( $Pr(R = t) = 1$ ), and we now show that  $\sigma(T_R) / \langle T_R \rangle \leq 1 \iff \langle T_{min} \rangle \geq \frac{1}{2} \frac{\langle \min(T, R)^2 \rangle}{\langle \min(T, R) \rangle}$  for every  $t \geq 0$  (regardless of its optimality). We do this to emphasize that Eq. (9) in the main text generalizes Eq. (8) to sharp, but not necessarily optimal, restart times.

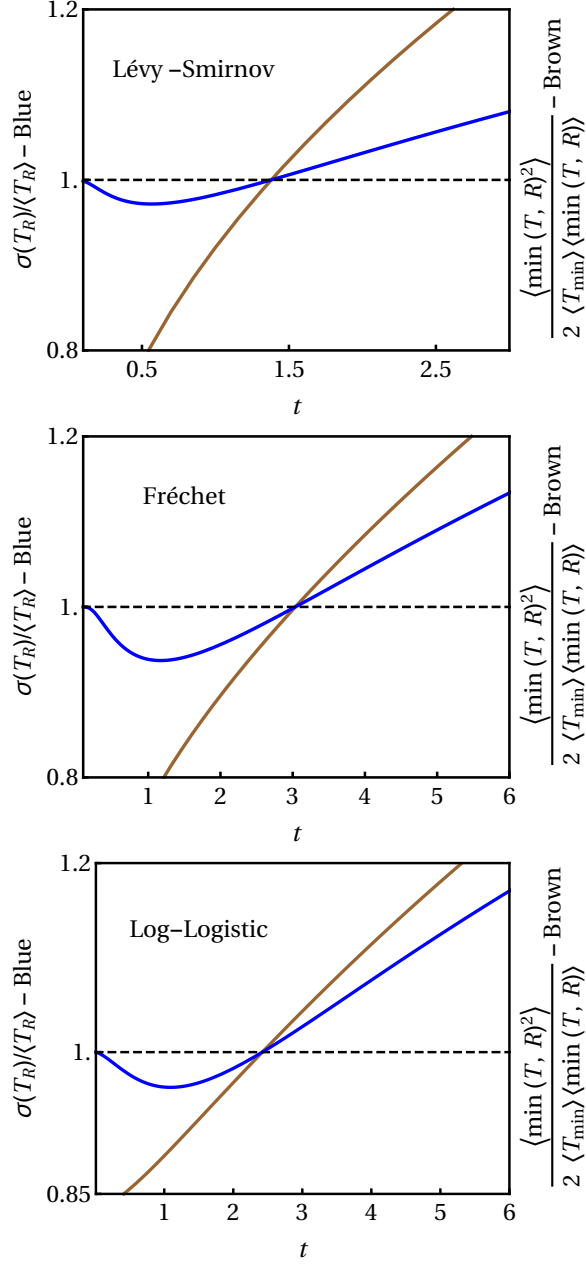


Figure S1: Plots demonstrating that Eq. (9) in the main text holds for all three cases considered in Fig. (5). From top to bottom: Lévy-Smirnov, Fréchet, Log-Logistic.

In Figure (S2), we take  $T$  as in Eq. (5.1) above and examine three different restart time distributions (Uniform, Gamma, Weibull) for  $R$ . We show that  $\sigma(T_R) / \langle T_R \rangle \leq 1 \iff \langle T_{\min} \rangle \geq \frac{1}{2} \frac{\langle \min(T, R)^2 \rangle}{\langle \min(T, R) \rangle}$  in all three cases and regardless of the optimality of mean restart time  $\langle R \rangle$ . We do this to emphasize that Eq. (9) in the main text generalizes Eq. (8) to any restart time distribution.

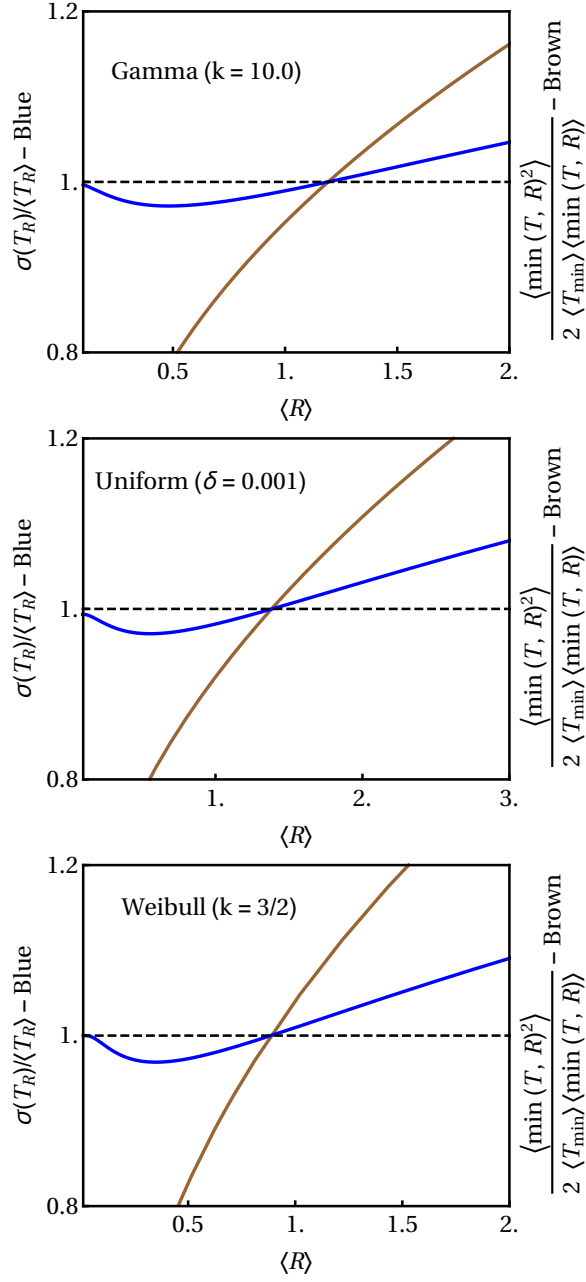


Figure S2: Plots demonstrating that Eq. (9) in the main text holds for restart time distributions other than the sharp. From top to bottom: (i) Gamma distribution  $f_R(t) = \frac{1}{\Gamma(k)\theta^k} t^{k-1} e^{-t/\theta}$  with  $k = 10$  and  $\theta = \langle R \rangle / k$ ; (ii) Uniform distribution  $f_R(t) = 1/\delta$  for  $t \in [\langle R \rangle - \delta/2, \langle R \rangle + \delta/2]$  and  $f_R(t) = 0$  otherwise; (iii) Weibull distribution  $f_R(t) = \frac{k}{\lambda} \left(\frac{t}{\lambda}\right)^{k-1} e^{-(t/\lambda)^k}$  with  $k = 3/2$  and  $\lambda = \langle R \rangle / \Gamma(5/3)$ .

## References

- [1] Gallager, R.G., 2013. Stochastic processes: theory for applications. Cambridge University Press.